Denotational Semantics

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Lectures for Part II CST 2026/2027

Recap of Lecture I

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- We looked for fixed points among partial functions
- We defined the semantics to be the least fixed point.

Domains and Fixed Points

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Posets and monotone functions

Partially ordered set

A partial order on a set D is a binary relation $\sqsubseteq$ that is

reflexive: $\forall d \in D.\ d \sqsubseteq d$

transitive: $\forall d, d', d'' \in D.\ d \sqsubseteq d' \sqsubseteq d'' \Rightarrow d \sqsubseteq d''$

antisymmetric: $\forall d, d' \in D.\ d \sqsubseteq d' \sqsubseteq d \Rightarrow d = d'$.

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A pair $(D, \sqsubseteq)$ is called a partially ordered set, or simply poset.

Example: Partial functions $X \rightharpoonup Y$

Set: Partial functions $f : X \rightharpoonup Y$, *i.e.*
Total functions $f : A \to Y$ where $A \subseteq X$

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Other examples:

- the powerset $\mathcal{P}(X)$ of any set X
- $\mathbb{N}, \mathbb{Z}, \ldots$ with the usual order $\leq$
- $\mathbb{N}$ with $x \sqsubseteq y$ when x divides y

The agrees with the order that we defined on $\mathrm{State} \rightharpoonup \mathrm{State}$ to give semantics for while loops.

Monotonicity

A function $f : D \to E$ between posets is monotone if it preserves the order:

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Exercise: Check that the function $F_{b,c}$ is monotone.

Domains and Fixed Points

Least elements and least upper bounds

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Note that $\bot_S$ is by definition an element of S!

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- Lubs are also known as joins, supremums or limits
- The lub of S does *not* need to be in S!

Least upper bounds of chains

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The lub of a sequence does *not* need to be an element of the sequence!

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- We can discard elements: $\bigsqcup_n d_n = \bigsqcup_n d_{n+k}$ for any $k \in \mathbb{N}$
- If $d_k = d_{k+1} = d_{k+2} = \ldots$ for some $k \in \mathbb{N}$, then $\bigsqcup_{n \in \mathbb{N}} d_n = d_k$

Diagonalisation

Let D be a poset and $(d_{m,n})_{m,n \in \mathbb{N}}$ be an increasing doubly-indexed sequence in D:

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Then, assuming they exist, the lubs form two chains

$$\bigsqcup_{n\in\mathbb{N}} d_{0,n} \sqsubseteq \bigsqcup_{n\in\mathbb{N}} d_{1,n} \sqsubseteq \bigsqcup_{n\in\mathbb{N}} d_{2,n} \sqsubseteq \ldots$$

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Moreover, again assuming the lubs of these chains exist,

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Show the equality by proving that they are all lubs for the set $\{d_{m,n} : m, n \in \mathbb{N}\}$

Domains and Fixed Points

Complete partial orders and domains

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We will see that these are the ingredients we need to construct least fixed points.

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Equivalently, the partial function f has graph the union of the graphs of f_n.

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Beware: this definition of $\bigsqcup_n f_n$ gives a partial function only if the f_n form a chain!

Example: Finite cpos

Every finite poset is a cpo. Why?

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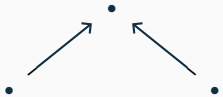
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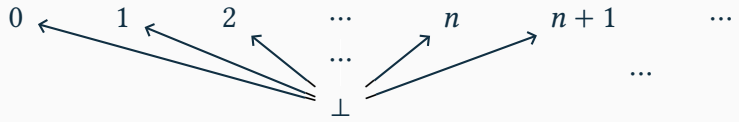
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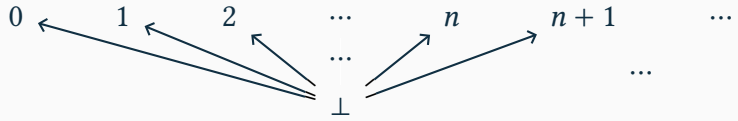
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Example: The flat natural numbers $\mathbb{N}_\perp$



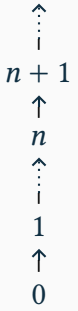
Example: The flat natural numbers $\mathbb{N}_\bot$



Every chain in $\mathbb{N}_\bot$ is eventually constant.

Example: Vertical natural numbers

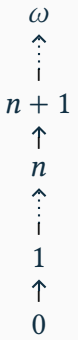
Is $(\mathbb{N}, \leq)$ a domain?



No! (Why?)

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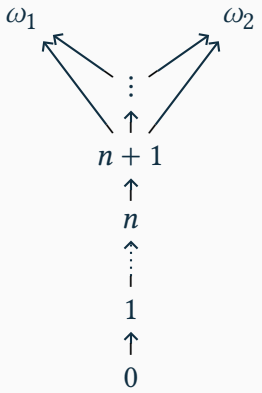
What if we add a greatest element?



Yes!

Example: Vertical natural numbers

What if we add two greatest elements?



No! (Why?)